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A PID CONTROLLER FOR THE ROBUST STABILIZATION OF SISO LINEAR SYSTEMS*

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Abstract — This work deals with the problem of the robust stabilization of minimum phase single-input single-output (SISO) systems. A PID controller is proposed for their stabilization, under the assumption that the relative degree and the sign of the high frequency gain are known.

A PID CONTROLLER

The purpose of this note is a modification of the classical stabilization theory for minimum phase single-input single-output (SISO) linear systems to point out the need of using an integrator to reject the effects of the entire dynamics of the system (assumed to be completely unknown) so as to naturally yield a PID (proportional-integral-derivative) controller. The PID controllers have been studied for a long period by several authors (see e.g., [1–4]), under the assumption that a knowledge of the system is available and that an observer can be designed for the estimation of the state variables, which are not directly measurable, thus yielding a multi-integrator controller.

In this note, with reference to minimum phase SISO systems, we will show how a PID controller (with a single-integrator) can be designed also in the case of completely unknown dynamics.

Let us consider a SISO linear system described by the following transfer function:

$$\frac{y(s)}{u(s)} = G(s) := H\,\frac{b(s)}{a(s)} \equiv H\,\frac{b_0 + b_1\,s + \cdots + b_{n-r-1}\,s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \cdots + a_{n-1}\,s^{n-1} + s^n}, \tag{1}$$

where: n is the number of poles, r is the relative degree, H is the system gain at high frequencies, u and y are the input and output variables, respectively, $b(s) := b_0 + b_1\,s + \cdots + b_{n-r-1}\,s^{n-r-1} + s^{n-r}$, and $a(s) := a_0 + a_1\,s + \cdots + a_{n-1}\,s^{n-1} + s^n$.

ASSUMPTIONS.

(a.1) The relative degree r is known.

(a.2) The transfer function $G(s)$ is minimum phase, i.e., the spectrum of the polynomial $b(s)$ is in the open left-half plane.

(a.3) The sign $\sigma(H)$ of the high frequencies gain H is known.

(a.4) The measure of the output variable $y(t)$ and of its time derivatives $y^{(i)}(t)$ of order i is available up to order $r-1$.

(a.5) The numerator and the denominator of the transfer function $G(s)$ are relative prime and the unobservable and uncontrollable modes are asymptotically stable.

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REMARKS.

(r.1) The number n of poles is unknown and can be arbitrarily large.

(r.2) The values of coefficients $a_i, i = 0, \dots, n-1$, $b_i, i = 0, \dots, n-r-1$, and H are unknown.

(r.3) The number of the unstable poles of $G(s)$ is unknown.

(r.4) In general (see e.g., [5] in the recent literature), the number of unstable poles is assumed to be partially known (the so-called *stable-factor* uncertainty) and the uncertainties acting on the plant to be stabilized are assumed to be additive or multiplicative.

Under the Assumption (a.5), a minimal realization of $G(s)$ is given by:

$$\begin{aligned} \dot{x} &= A\,x + B\,u, \\ y &= C\,x, \end{aligned} \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ H \end{bmatrix},$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-r-1} & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

It is well known that, the normal form of system (2) can be obtained by defining the following new coordinates:

$$\begin{aligned} z_i &:= C\,A^{i-1}\,x, & i &= 1, \dots, r, \\ w_i &:= x_i, & i &= 1, \dots, n-r. \end{aligned} \tag{3}$$

In the new coordinates, we obtain the system in the normal form, which has the following structure:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \qquad i = 1, \dots, r-1, \\ \dot{z}_r &= -c_0\,z_1 - \cdots - c_{r-1}\,z_r - d_0\,w_1 - \cdots - d_{n-r-1}\,w_{n-r} + H\,u \\ &\equiv -\sum_{i=0}^{r-1} c_i\,z_{i+1} - \sum_{i=0}^{n-r-1} d_i\,w_{i+1} + H\,u \\ \dot{w}_i &= w_{i+1}, \qquad i = 1, \dots, n-r-1, \\ \dot{w}_{n-r} &= -b_0\,w_1 - \cdots - b_{n-r-1}\,w_{n-r} + z_1 \\ &\equiv -\sum_{i=0}^{n-r-1} b_i\,w_{i+1}\,z_1, \\ y &= z_1. \end{aligned} \tag{4}$$

The equation describing the dynamics of $\dot{z}_r$ can be rewritten as follows:

$$\dot{z}_r = f(z_1, \dots, z_r, w_1, \dots, w_{n-r}, u) + u, \tag{5}$$

where:

$$f(z_1, \dots, z_r, w_1, \dots, w_{n-r}, u) := -\sum_{i=0}^{r-1} c_i\,z_{i+1} - \sum_{i=0}^{n-r-1} d_i\,w_{i+1} + (H-1)\,u.$$

Under the Assumption (a.2) and that term f is available, i.e., all the state vector is measurable and the values of coefficients $c_i, i = 0, \dots, r-1$, $d_i, i = 0, \dots, n-r-1$, and H are known, a possible control law for the stabilization of (4) is:

$$u := -h_0\,z_1 - h_1\,z_2 - \cdots - h_{r-1}\,z_r - f \equiv -\sum_{i=0}^{r-1} h_i\,z_{i+1} - f, \tag{6}$$

which guarantees that the eigenvalues of the closed loop system (4), (6) coincide with the roots of $h(s)$ and $b(s)$, with $h(s) = h_0 + h_1 s + \cdots + h_{r-1} s^{r-1} + s^r$. Notice that, Equation (6) defines in an implicit way the control u which can be rendered explicit in a straightforward manner under the assumption $H \neq 0$:

$$u = -\frac{1}{H}\left(\sum_{i=0}^{r-1} h_i z_{i+1} - \sum_{i=0}^{r-1} c_i z_{i+1} - \sum_{i=0}^{n-r-1} d_i w_{i+1}\right).$$

Since all the state vector is not measurable and the values of coefficients $c_i, i = 0,\ldots,r-1$, d_i, $i = 0,\ldots,n-r-1$, and H are not known, control law (6) cannot be actually implemented. A possible way to overcome this difficulty is the use of an integral action to reject the influence of term f, as in the following PID control law which is proposed here for the stabilization of the linear system (4):

$$u := -h_0 z_1 - h_1 z_2 - \cdots - h_{r-1} z_r - \hat{f} \equiv -\sum_{i=0}^{r-1} h_i z_{i+1} - \hat{f}, \tag{7}$$

where

$$\hat{f} = \xi + k z_r, \tag{8}$$
$$\dot{\xi} = -k\xi - k^2 z_r - k u, \tag{9}$$

$h_i, i = 0,\ldots,r-1$, are suitable positive constants, and $k := \sigma(H)\mu$, with μ being a suitable positive constant.

REMARK.

(r.5) Notice that the realization of the dynamic controller (7)–(9) requires the knowledge of the relative degree r, the measure of $z_i(t)$, $i = 1,\ldots,r$ (i.e., the measure of $y^{(i)}(t)$, $i = 0,\ldots,r-1$) and the sign of parameter H. Hence, under the Assumptions (a.1), (a.3) and (a.4), the control law (7)–(9) is well defined.

ASSUMPTION.

(a.6) The parameters h_i, $i = 0,\ldots,r-1$, are chosen so that the spectrum of the polynomial $h(s) := h_0 + h_1 s + \cdots + h_{r-1} s^{r-1} + s^r$ is in the open left-half plane.

Taking the Laplace transform of Equations (7)–(9), we find the equations of the proposed controller in the s domain:

$$u(s) = -K(s)\, y(s), \tag{10}$$
$$K(s) := h(s) - s^r + \mu\,\sigma(H)\,\frac{h(s)}{s}. \tag{11}$$

The main result of this paper regarding the stability properties of the closed loop system (4), (7)–(9), (or equivalently (1), (10), (11)) can be stated and proved as follows.

THEOREM 1. *Under the Assumptions (a.1)–(a.6), there exists a constant value μ^* such that the closed loop system (4), (7)–(9) is asymptotically stable for any $\mu \geq \mu^*$.*

REMARK.

(r.6) In view of Theorem 1, the controller (10) and (11) has the following property: the transfer function of the closed loop system (1), (10), (11),

$$T_\mu(s) := \frac{K(s)\, H\, \dfrac{a(s)}{b(s)}}{1 + K(s)\, H\, \dfrac{a(s)}{b(s)}} =: \frac{N_\mu(s)}{D_\mu(s)}, \tag{12}$$

with

$$N_\mu(s) := \mu\,|H|\,b(s)\left(h(s) + \frac{1}{\mu}\,\sigma(H)\,s\,(h(s) - s^r)\right) \qquad \text{and}$$
$$D_\mu(s) := s\,a(s) + \mu\,|H|\,b(s)\left(h(s) + \frac{1}{\mu}\,\sigma(H)\,s\,(h(s) - s^r)\right),$$

is asymptotically stable for sufficiently high values of parameter μ.

PROOF OF THEOREM 1. Since there exists a polynomial $\delta(s)$ having degree $\deg(\delta(s)) \le n$ such that $a(s) = b(s)\,h(s) + \delta(s)$, the denominator $D_\mu(s)$ of the transfer function (12) can be rewritten as follows:

$$D_\mu(s) = (s + \mu\,|H|)\;b(s)\,h(s) + q(s), \tag{13}$$

with polynomial $q(s) := s\,\delta(s) + H\,s\,(h(s) - s^r)\,b(s)$ having degree $\deg(q(s)) \le n$. If $D_\mu(\overline{s}) = 0$ with $\mathrm{Re}(\overline{s}) \ge 0$, then

$$\overline{s} = -\frac{q(\overline{s})}{b(\overline{s})\,h(\overline{s})} - \mu\,|H|$$

and therefore:

$$-\,\mathrm{Re}\,\frac{q(\overline{s})}{b(\overline{s})\,h(\overline{s})} \ge \mu\,|H|. \tag{14}$$

It is noted that, in view of Assumptions (a.2) and (a.6), $b(\overline{s}) \ne 0$ and $h(\overline{s}) \ne 0$ for any $\overline{s}$ such that $\mathrm{Re}(\overline{s}) \ge 0$.

Equation (14) implies that

$$\left|\frac{q(\overline{s})}{b(\overline{s})\,h(\overline{s})}\right| \ge \mu\,|H|. \tag{15}$$

Since

$$\frac{q(s)}{b(s)\,h(s)}$$

is proper and asymptotically stable (i.e., $\deg\,(q(s)) \le \deg\,(b(s)\,h(s))$ and $b(\overline{s}) \ne 0$, $h(\overline{s}) \ne 0$ for any $\overline{s}$ such that $\mathrm{Re}\,(\overline{s}) \ge 0$), we have that

$$\sup_{\mathrm{Re}(\overline{s}) \ge 0} \left|\frac{q(\overline{s})}{b(\overline{s})\,h(\overline{s})}\right|$$

is finite and independent of μ. Hence Equation (15) cannot be satisfied for high values of positive parameter μ, thus proving the existence of a positive μ^* such that the closed loop system is asymptotically stable for any $\mu \ge \mu^*$. ∎

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